as minus the end restraint forces due to the loads acting in the element, when the ends are held fixed. The equivalence between this definition and that given by Tong from the variation of Eq. (2) can easily be shown by means of the virtual work equation where the virtual displacements are taken as the actual displacements in the element when one of the end displacements, $q_i$, is equal to one and all others are zero:

$$\frac{\partial}{\partial q_i}\left[\int_{x_i}^{x_{i+1}} pw dx\right] = \int_{x_i}^{x_{i+1}} p\delta w \, dx = \int_{x_i}^{x_{i+1}} G\delta w dx + Q_i$$

where $G$ is the vector of internal forces associated with the degrees of freedom of the problem (a function of $x$) due to the element loads for fixed end condition, and $Q_i$ is the generalized force associated with $q_i$. Since $G$ constitutes a state of stress that corresponds to a fixed end condition, it immediately follows that the first term in the right-hand side of the preceding equation vanishes.

The validity of Eq. (8) can also be proved without reference to variational considerations by pointing out that the internal forces $G$ satisfy equilibrium and compatibility inside each element and that the interelement compatibility and displacement boundary conditions are satisfied by definition of the generalized displacements; thus it only remains to insure equilibrium at the element ends and to satisfy the force boundary conditions that may be prescribed, which is done through Eq. (8). Finally, it may be concluded in agreement with Tong, that the solution of Eq. (8) for the nodal displacements is exact because both **K** and **Q** are exact.

### References

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## Comment on "Heat-Transfer Characteristies of Hot-Gas Ignition"

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### Nomenclature

$D_i$ = igniter nozzle throat diameter
$D_p$ = test-section diameter
$p_0$ = stagnation pressure upstream nozzle
$p_w$ = static pressure in the dead-air region
$\bar{p}_w$ = $p_w/p_0$
$x$ = coordinate along test section
$\bar{x}$ = $x/D_p$

When investigating heat transfer during head-end hot-gas ignition, the authors[1] distinguished only two cases of flow pattern (Figs. 1a and 1d). One case corresponds to the reattachment of supersonic jet (Fig. 1a) and the other one has no reattachment at all (Fig. 1d). This is an over-simplification because there are two other intermediate cases.[2] One is significant for subsonic reattachment (Fig. 1c) and the other for mixed reattachment and oscillating flow (Fig. 1b). The type of flow appearing depends on the level of nondimensional pressure $\bar{p}_w$ in the dead-air region; that depends, however, on over-all pressure ratio and nozzle area to test-section area ratio. The heat transfer might be influenced very significantly by the different location or oscillating motion of the reattachment region. In my experiments the subsonic reattachment line defined by oil-film technique was located much further from nozzle $\bar{x} = 1.8$ for $D_p/D_i = 1.58$ than the supersonic one, $\bar{x} = 0.33$. The cyclic oscillation of the reattachment region is associated with strong pressure and shock-wave cyclic oscillation. The change of flow pattern is also cyclic, and therefore the wall is being touched by the supersonic and subsonic stream in turn. The oscillation is self-excited, and for each value of $D_p/D_i$ it appears in a definite range of $\bar{p}_w$. These and some other features of the oscillating flow are described in detail in Refs. 2 and 3. It is therefore of interest to know how exact was the coincidence of reattachment line location with the heat-transfer maximum and to what types of reattachment the results given in Figs. 8–10 correspond.

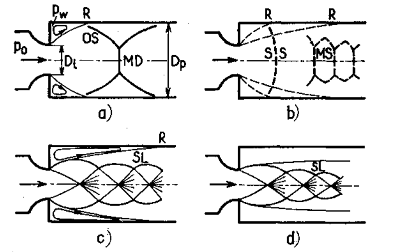


**Fig. 1 Jet expansion from sonic convergent nozzles; R = reattachment line, OS = oblique shock, MD = Mach disk, SS = single shock, MS = multishock, SL = subsonic layer.**

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[2] Jungowski, W. M., "Investigation of Flow Pattern, Boundary Conditions and Oscillation Mechanism in a Compressible Flow through Sudden Enlargement of a Duct," Warsaw Technical University Publications, Mechanika No. 3, 1968.

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## Reply by Author to W. M. Jungowski

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In answer to Professor Jungowski's first question, the point of reattachment was not determined during the experimental studies. However, the expanding jet correlations of Love et al. (Ref. 10 of the paper) were used to compute the point where a constant pressure jet boundary would intersect the wall. Of course, the actual point of reattachment is located somewhere in the shear layer encompassing this jet boundary. Also, the jet boundary predictions are based on the existence of constant pressure when, in actuality, the pressure rises as the shear layer approaches the wall and recompression begins. Nevertheless, it has been found that the predicted point of intersection of the constant pressure jet boundary with the wall does coincide with the point of maximum heat transfer to within the scale of the measurements (spacing between thermocouples was $\frac{1}{4}$ of a duct diameter).

The flow cases suggested by Professor Jungowski appear to be reasonable. Perhaps the transition from one flow regime to another furnishes an explanation of the changing character of the heat transfer dependence on the port-to-exhaust nozzle throat area ratio evidenced in Figs. 8–10 of the paper. Since no detailed measurements were made in the jet itself, it is not possible to ascertain the flow regime prevailing for each test.

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